Discrete Time Fourier Transform (DTFT)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega \quad \text{and} \quad X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

DTFT Properties and Pairs		
$x[-n]$	$\overset{DTFT}{\Longleftrightarrow}$	$X\left(e^{-j\omega}\right)$
$x^*[n]$	$\overset{DTFT}{\Longleftrightarrow}$	$X^*\left(e^{-j\omega}\right)$
$x[n-n_0]$	$\overset{DTFT}{\Longleftrightarrow}$	$X\left(e^{j\omega}\right)e^{-j\omega n_0}$
$x[n]e^{j\omega_0 n}$	$\overset{DTFT}{\Longleftrightarrow}$	$X\left(e^{j(\omega-\omega_0)}\right)$
$ax_1[n] + bx_2[n]$	$\overset{DTFT}{\Longleftrightarrow}$	$aX_1\left(e^{j\omega}\right) + bX_2\left(e^{j\omega}\right)$
$x_1[n]x_2[n]$	$\overset{DTFT}{\Longleftrightarrow}$	$\frac{1}{2\pi}X_1\left(e^{j\omega}\right) * X_2\left(e^{j\omega}\right)$
$x_1[n] * x_2[n]$	$\overset{DTFT}{\Longleftrightarrow}$	$X_1\left(e^{j\omega}\right)X_2\left(e^{j\omega}\right)$
$\delta[n]$	$\overset{DTFT}{\Longleftrightarrow}$	1
$\delta[n-n_0]$	$\overset{DTFT}{\Longleftrightarrow}$	$e^{-j\omega n_0}$
1	$\overset{DTFT}{\Longleftrightarrow}$	$2\pi\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$
$e^{j\omega_0 n}$	$\overset{DTFT}{\Longleftrightarrow}$	$2\pi\sum_{k=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi k)$
$u[n] = \begin{cases}1 & \text{if } n \geq 0\\ 0 & \text{otherwise}\end{cases}$	$\overset{DTFT}{\Longleftrightarrow}$	$\frac{1}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\overset{DTFT}{\Longleftrightarrow}$	$\frac{1}{1-ae^{-j\omega}}$
$\sin[\omega_0 n + \theta]$	$\overset{DTFT}{\Longleftrightarrow}$	$\sum_{k=-\infty}^{+\infty}\frac{\pi}{j}\left[e^{j\theta}\delta(\omega-\omega_0-2\pi k) - e^{-j\theta}\delta(\omega+\omega_0-2\pi k)\right]$
$\cos[\omega_0 n + \theta]$	$\overset{DTFT}{\Longleftrightarrow}$	$\sum_{k=-\infty}^{+\infty}\pi\left[e^{j\theta}\delta(\omega-\omega_0-2\pi k) + e^{-j\theta}\delta(\omega+\omega_0-2\pi k)\right]$
$x[n] = \begin{cases}1 & 0 \leq n \leq M\\ 0 & \text{otherwise}\end{cases}$	$\overset{DTFT}{\Longleftrightarrow}$	$\frac{\sin\left[\omega\left(\frac{M+1}{2}\right)\right]}{\sin(\omega/2)}e^{-j\omega\frac{M}{2}}$
$\frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$	$\overset{DTFT}{\Longleftrightarrow}$	$\sum_{k=-\infty}^{+\infty}\text{rect}\left(\frac{\omega-2\pi k}{2\omega_c}\right)$

Note:

$$\text{rect}(x) = \begin{cases}1 & \lvert x\rvert \leq \frac{1}{2}\\ 0 & \text{otherwise}\end{cases}, \qquad \text{sinc}(x) = \frac{\sin(x)}{x}$$

Euler's Formula:

$$e^{jx} = \cos(x) + j\sin(x)$$

$$\cos(x) = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$$

$$\sin(x) = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)$$

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(jinsol.lee@gatech.edu)

2D Discrete Time Fourier Transform (2D DTFT)

$$F(\omega_1,\omega_2) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]e^{-j(\omega_1 m+\omega_2 n)}$$

$$f[m,n] = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(\omega_1,\omega_2)e^{j(\omega_1 m+\omega_2 n)}d\omega_1 d\omega_2$$

2D DTFT Properties and Pairs		
$f[m-k,n-l]$	$\overset{DTFT}{\Longleftrightarrow}$	$e^{-j(k\omega_1+l\omega_2)}F(\omega_1,\omega_2)$
$f[m,n]e^{j2\pi(sm+tn)}$	$\overset{DTFT}{\Longleftrightarrow}$	$F(\omega_1-s,\omega_2-t)$
$f^*[m,n]$	$\overset{DTFT}{\Longleftrightarrow}$	$F^*(-\omega_1,-\omega_2)$
$f[-m,n]$	$\overset{DTFT}{\Longleftrightarrow}$	$F(-\omega_1,\omega_2)$
$f[m,-n]$	$\overset{DTFT}{\Longleftrightarrow}$	$F(\omega_1,-\omega_2)$
$f[m,n]*h[m,n]$	$\overset{DTFT}{\Longleftrightarrow}$	$F(\omega_1,\omega_2)H(\omega_1,\omega_2)$
$f[m,n]w[m,n]$	$\overset{DTFT}{\Longleftrightarrow}$	$\frac{1}{4\pi^2}F(\omega_1,\omega_2)*W(\omega_1,\omega_2)$
$f[n,m]$	$\overset{DTFT}{\Longleftrightarrow}$	$F(\omega_2,\omega_1)$
$f[n]g[m]$	$\overset{DTFT}{\Longleftrightarrow}$	$F(\omega_1)G(\omega_2)$
$f[am+bn,cm+dn]$	$\overset{DTFT}{\Longleftrightarrow}$	$\begin{cases} F(d\omega_1-c\omega_2,-b\omega_1+a\omega_2) & \text{if } ad-bc=1 \\ F(-d\omega_1+c\omega_2,b\omega_1-a\omega_2) & \text{if } ad-bc=-1 \end{cases}$
$\delta[m,n]$	$\overset{DTFT}{\Longleftrightarrow}$	1
1	$\overset{DTFT}{\Longleftrightarrow}$	$4\pi^2\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\delta(\omega_1-2\pi k_1,\omega_2-2\pi k_2)$
$x[m,n]=\begin{cases}1 & \text{if } 0\le m\le M\ ,\ 0\le n\le N \\ 0 & \text{otherwise}\end{cases}$	$\overset{DTFT}{\Longleftrightarrow}$	$\frac{\sin\left[\omega_1(\frac{M+1}{2})\right]}{\sin(\omega_1/2)}\frac{\sin\left[\omega_2(\frac{N+1}{2})\right]}{\sin(\omega_2/2)}e^{-j(\omega_1\frac{M}{2}+\omega_2\frac{N}{2})}$
$\sin[c_1m+c_2n]$	$\overset{DTFT}{\Longleftrightarrow}$	$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\frac{\pi}{j}\Big[\delta(\omega_1-c_1-2\pi k_1,\omega_2-c_2-2\pi k_2) - \delta(\omega_1+c_1-2\pi k_1,\omega_2+c_2-2\pi k_2)\Big]$
$\cos[c_1m+c_2n]$	$\overset{DTFT}{\Longleftrightarrow}$	$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\pi\Big[\delta(\omega_1-c_1-2\pi k_1,\omega_2-c_2-2\pi k_2) + \delta(\omega_1+c_1-2\pi k_1,\omega_2+c_2-2\pi k_2)\Big]$

Note:

Most of the 2D DTFT pairs can be obtained using the separability property and 1D DTFT.

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2D Discrete Time Fourier Transform (2D DTFT)

$$F(\mu,\nu) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi(\mu m+\nu n)} \quad \text{and} \quad f[m,n] = \int_{-0.5}^{0.5}\int_{-0.5}^{0.5} F(\mu,\nu)e^{j2\pi(\mu m+\nu n)}d\mu d\nu$$

2D DTFT Properties and Pairs

$f[m-k,n-l]$	$\overset{DTFT}{\Longleftrightarrow}$	$e^{-j2\pi(k\mu+l\nu)}F(\mu,\nu)$
$f[m,n]e^{j2\pi(sm+tn)}$	$\overset{DTFT}{\Longleftrightarrow}$	$F(\mu-s,\nu-t)$
$f^*[m,n]$	$\overset{DTFT}{\Longleftrightarrow}$	$F^*(-\mu,-\nu)$
$f[-m,n]$	$\overset{DTFT}{\Longleftrightarrow}$	$F(-\mu,\nu)$
$f[m,-n]$	$\overset{DTFT}{\Longleftrightarrow}$	$F(\mu,-\nu)$
$f[m,n]*h[m,n]$	$\overset{DTFT}{\Longleftrightarrow}$	$F(\mu,\nu)H(\mu,\nu)$
$f[m,n]w[m,n]$	$\overset{DTFT}{\Longleftrightarrow}$	$F(\mu,\nu)*W(\mu,\nu)$
$f[n,m]$	$\overset{DTFT}{\Longleftrightarrow}$	$F(\nu,\mu)$
$f[n]g[m]$	$\overset{DTFT}{\Longleftrightarrow}$	$F(\mu)G(\nu)$
$f[am+bn,cm+dn]$	$\overset{DTFT}{\Longleftrightarrow}$	$\begin{cases} F(d\mu-c\nu,-b\mu+a\nu) & \text{if } ad-bc=1 \\ F(-d\mu+c\nu,b\mu-a\nu) & \text{if } ad-bc=-1 \end{cases}$
$\delta[m,n]$	$\overset{DTFT}{\Longleftrightarrow}$	1
1	$\overset{DTFT}{\Longleftrightarrow}$	$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\delta(\mu-k_1,\nu-k_2)$
$x[m,n] = \begin{cases} 1 & \text{if } 0 \le m \le M\ ,\ 0 \le n \le N \\ 0 & \text{otherwise} \end{cases}$	$\overset{DTFT}{\Longleftrightarrow}$	$\frac{\sin(\pi\mu(M+1))}{\sin(\pi\mu)}\frac{\sin(\pi\nu(N+1))}{\sin(\pi\nu)}e^{-j2\pi(\mu\frac{M}{2}+\nu\frac{N}{2})}$
$\sin[c_1m+c_2n]$	$\overset{DTFT}{\Longleftrightarrow}$	$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\frac{1}{2j}\Big[\delta(\mu-c_1-k_1,\nu-c_2-k_2) - \delta(\mu+c_1-k_1,\nu+c_2-k_2)\Big]$
$\cos[c_1m+c_2n]$	$\overset{DTFT}{\Longleftrightarrow}$	$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\frac{1}{2}\Big[\delta(\mu-c_1-k_1,\nu-c_2-k_2) + \delta(\mu+c_1-k_1,\nu+c_2-k_2)\Big]$

Discrete Fourier Transform (DFT)

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}} \quad \text{and} \quad X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$$

DFT Properties and Pairs

$x\left[((n-m))_N\right]$	$X[k]e^{-j\frac{2\pi}{N}km}$
$x_1[n]x_2[n]$	$\frac{1}{N}X_1[k] \circledast X_2[k]$
$x_1[n] \circledast x_2[n]$	$X_1[k]X_2[k]$
$x^*[n]$	$X^*\left[((-k))_N\right]$
$x^*\left[((-n))_N\right]$	$X^*[k]$
$\mathrm{Re}\{x[n]\}$	$\frac{1}{2}\left\{X\left[((k))_N\right] + X\left[((-k))_N\right]\right\}$
$j\mathrm{Im}\{x[n]\}$	$\frac{1}{2}\left\{X\left[((k))_N\right] - X\left[((-k))_N\right]\right\}$
$x[n] \quad$ (period N)	$X_N[k] \quad$ (N point DFT)
$\sum_{k=-\infty}^{+\infty}\delta[n+Nk] = \begin{cases}1 & \text{if } n = 0, \pm N, \pm 2N, \cdots \\ 0 & \text{otherwise}\end{cases}$	$1 \quad$ (period N)
$1 \quad$ (period N)	$\mathrm{N}\sum_{m=-\infty}^{+\infty}\delta[k+Nm] = \begin{cases}1 & \text{if } k = 0, \pm N, \pm 2N, \cdots \\ 0 & \text{otherwise}\end{cases}$
$e^{\frac{2\pi}{N}k_0 n}$	$N\delta\left[((k-k_0))_N\right]$
$\cos\left(\frac{2\pi}{N}k_0 n\right)$	$\frac{N}{2}\left\{\delta\left[((k-k_0))_N\right] + \delta\left[((k+k_0))_N\right]\right\}$

2D-Discrete Fourier Transform (2D DFT)

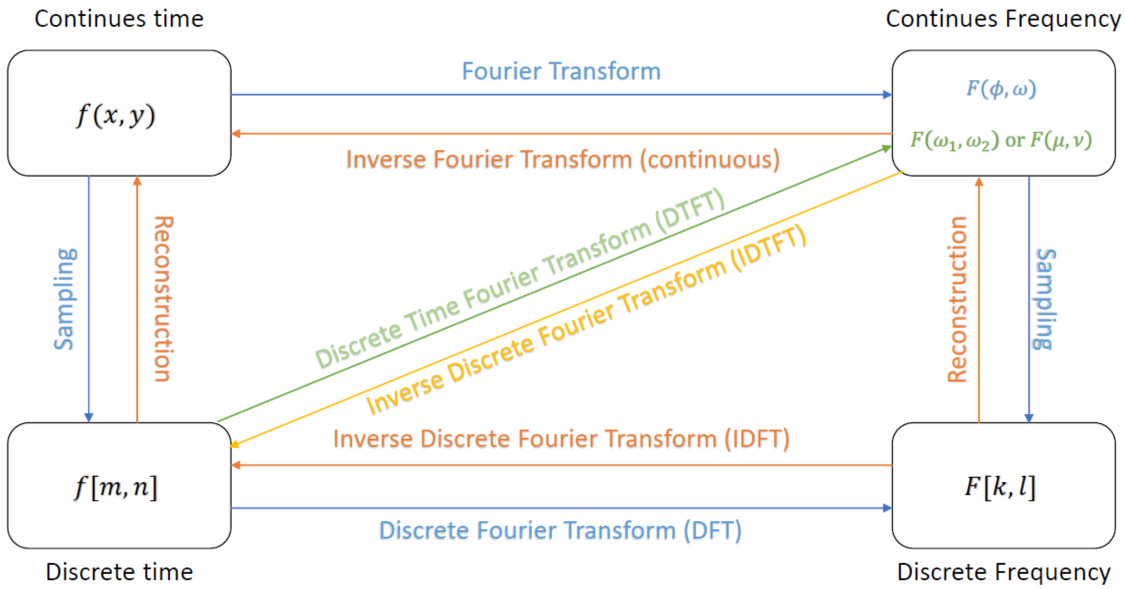
$$F[k,l] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j\frac{2\pi}{M}km - j\frac{2\pi}{N}ln} \quad \begin{cases}0 \le k \le M-1 \\ 0 \le l \le N-1\end{cases}$$

$$f[m,n] = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F[k,l]e^{j\frac{2\pi}{M}km + j\frac{2\pi}{N}ln} \quad \begin{cases}0 \le m \le M-1 \\ 0 \le n \le N-1\end{cases}$$

2D DFT Properties

$af[m,n] + bg[m,n]$	$aF[k,l] + bG[k,l]$
$f[((m-m_0))_M, ((n-n_0))_N]$	$W_M^{m_0 k}W_N^{n_0 l}F[k,l]\left(W_N = e^{-j\frac{2\pi}{N}}\right)$
$f[m,n]$ real	$F^*[k,l] = F[((M-k))_M), ((N-l))_N]$
$f_s[m,n] = \frac{1}{2}[f[m,n] + f^*[((M-m))_M, ((N-n))_N]]$	$\frac{1}{2}\{F[k,l] + F^*[k,l]\} = \mathrm{Re}\{F[k,l]\}$
$f_a[m,n] = \frac{1}{2}[f[m,n] - f^*[((M-m))_M, ((N-n))_N]]$	$\frac{1}{2}\{F[k,l] - F^*[k,l]\} = j\mathrm{Im}\{F[k,l]\}$
$f[n,m]$	$F[l,k]$
$f\left[((M-m))_M, n\right]$	$F\left[((M-k))_M, l\right]$
$f\left[m, ((N-n))_N\right]$	$F\left[k, ((N-l))_N\right]$
$f\left[((M-m))_M, ((N-n))_N\right]$	$F\left[((M-k))_M, ((N-l))_N\right]$

DFT/DTFT flowchart



Linear Convolution

- 1-Dimensional:

$$g[m] = f[m] * h[m] = \sum_{k=-\infty}^{+\infty} f[k]h[m-k]$$

- 2-Dimensional:

$$g[m,n] = f[m,n] * h[m,n] = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} f[k,l]h[m-k,n-l]$$

Circular Convolution

- 1-Dimensional, $M-$point:

$$g[m] = f[m] \circledast h[m] = \sum_{k=0}^{M-1} f[k]h[((m-k))_M]$$

- 2-Dimensional, $M \times N-$point:

$$g[m,n] = f[m,n] \circledast h[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f[k,l]h[((m-k))_M, ((n-l))_N]$$

Sampling and reconstruction

- 1-Dimensional:

$$f[m] = f_a(mT)$$

$$f_a(t) = \sum_{m=-\infty}^{+\infty} f[m] \frac{\sin\left(\frac{\pi}{T}(t-mT)\right)}{\frac{\pi}{T}(t-mT)}$$

- 2-Dimensional:

$$f[m,n] = f_a(mX, nY)$$

$$f_a(x,y) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} f[m,n] \frac{\sin\left(\frac{\pi}{X}(x-mX)\right)}{\frac{\pi}{X}(x-mX)} \frac{\sin\left(\frac{\pi}{Y}(y-nY)\right)}{\frac{\pi}{Y}(y-nY)}$$

2D Discrete Cosine Transforms (2D DCT)

$$F_c[k,l] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} 4f[m,n] \cos\left[\frac{\pi}{M}k(m+\frac{1}{2})\right] \cos\left[\frac{\pi}{N}l(n+\frac{1}{2})\right]$$

$$f[m,n] = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} c[k]c[l]F_c[k,l] \cos\left[\frac{\pi}{M}k(m+\frac{1}{2})\right] \cos\left[\frac{\pi}{N}l(n+\frac{1}{2})\right]; \text{ where } c[k] = \begin{cases} \frac{1}{2} & k=0 \\ 1 & k \neq 0 \end{cases}$$

Wavelet transforms

For a given 2D matrix, $\mathbf{X} \in \mathbb{R}^{2^n \times 2^n}$, the wavelet transform is given by:

$$Y = \frac{1}{2^{2n}} \mathbf{H}_n \mathbf{X} \mathbf{H}_n^T$$

and the inverse transform is given by:

$$X = \mathbf{H}_n^T \mathbf{Y} \mathbf{H}_n$$

Where $\mathbf{H}_n \in \mathbb{R}^{2^n \times 2^n}$ is a transformation matrix. H_n is defined differently for different transforms and can computed using H_{n-1} as follows:

- Haar transform:
$$\mathbf{H}_n = \begin{bmatrix} \mathbf{H}_{n-1} \otimes [1, 1] \\ \sqrt{2^{n-1}} I_{2^{n-1}} \otimes [1, -1] \end{bmatrix}$$
  For $n = 1, 2, ...,$ where $I_{2^n-1} \in \mathbb{R}^{2^{n-1} \times 2^{n-1}}$ is the identity matrix and $\mathbf{H}_0 = 1$

- Hadamard transform:
$$\mathbf{H}_n = \mathbf{H}_1 \otimes \mathbf{H}_{n-1} = \begin{bmatrix} \mathbf{H}_{n-1} & \mathbf{H}_{n-1} \\ \mathbf{H}_{n-1} & -\mathbf{H}_{n-1} \end{bmatrix}$$
  For $n = 1, 2, ...,$ where $\mathbf{H}_0 = [\ 1\ ]$

- Walsh-Hadamard transform: The transformation matrix W_n is the similar to H_n for the Hadamard transform with the rows ordered in an ascending order according to the number of sign changes.

For a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{p \times q}$, the Kronecker product $A \otimes B \in \mathbb{R}^{mp \times nq}$ is defined as:

$$A \otimes B = \begin{bmatrix} a_{1,1}\mathbf{B} & a_{1,2}\mathbf{B} & \dots & a_{1,n}\mathbf{B} \\ a_{2,1}\mathbf{B} & \ddots & & a_{2,n}\mathbf{B} \\ \vdots & & \ddots & \vdots \\ a_{m,1}\mathbf{B} & a_{m,2}\mathbf{B} & \dots & a_{m,n}\mathbf{B} \end{bmatrix}$$

Image Quality Metrics

For two images $\mathbf{X} \in \mathbb{R}^{M \times N}$ and an its noisy approximation $\mathbf{Y} \in \mathbb{R}^{M \times N}$, we can compute the following quality metrics:

- Mean Squared Error (MSE):
$$\text{MSE} = \frac{1}{MN} \sum_{i=1}^{N} \sum_{i=1}^{M} (X[i,j] - Y[i,j])^2$$

- Peak Signal-to-Noise Ratio (PSNR):
$$\text{PSNR}(\mathbf{X}, \mathbf{Y}) = 20 \log_{10} \frac{\max \mathbf{X}}{\sqrt{\text{MSE}}}$$

- Structural Similarity (SSIM):

$$\begin{aligned}\text{SSIM}(\mathbf{X},\mathbf{Y}) &= [l(\mathbf{X},\mathbf{Y})]^{\alpha}\,[c(\mathbf{X},\mathbf{Y})]^{\beta}\,[s(\mathbf{X},\mathbf{Y})]^{\gamma} \\ &= \left[\frac{2\mu_{\mathbf{X}}\mu_{\mathbf{Y}}+C_1}{\mu_{\mathbf{X}}^2+\mu_{\mathbf{Y}}^2+C_1}\right]^{\alpha}\left[\frac{2\sigma_{\mathbf{X}}\sigma_{\mathbf{Y}}+C_2}{\sigma_{\mathbf{X}}^2+\sigma_{\mathbf{Y}}^2+C_2}\right]^{\beta}\left[\frac{\sigma_{\mathbf{XY}}+C_3}{\sigma_{\mathbf{X}}\sigma_{\mathbf{Y}}+C_3}\right]^{\gamma}\end{aligned}$$

where,
$\mu_{\mathbf{X}}, \mu_{\mathbf{Y}}$ are the mean values of $\mathbf{X}$ and $\mathbf{Y}$, respectively
$\sigma_{\mathbf{X}}, \sigma_{\mathbf{Y}}$ are the standard deviation values of $\mathbf{X}$ and $\mathbf{Y}$, respectively
$\sigma_{\mathbf{XY}}$ is the covariance between $\mathbf{X}$ and $\mathbf{Y}$. C_1, C_2, and C_3 are constants.

Histogram Equalization

For an image of size $M \times N$ whose pixels are in the range $[a, b]$, and n_k is the number of pixels with intensity value r_k. The transformation is defined as:

$$T_k = \frac{b-a}{MN}\sum_{j=1}^{k} n_k$$

Linear Filtering

- Gaussian smoothing:

$$g[m,n] = \frac{1}{2C}\sum_{i,j} f[k,l]\exp\left(-\frac{1}{2}\left[\frac{(m-k)^2}{\sigma_m^2}+\frac{(n-l)^2}{\sigma_n^2}\right]\right)$$

C is normalization constant.

- Spatial mean filtering:

$$g[m,n] = \left[\frac{\sum\limits_{[k,l]\in S_{mn}} w[k,l]\,(f[k,l])^p}{\sum\limits_{[k,l]\in S_{mn}} w[k,l]}\right]^{\frac{1}{p}}$$

For a given region in the image, S_{mn}. For $p = 1$, it is the arithmetic mean filter.

- Bilateral filtering:

$$g[m,n] = \frac{1}{2C}\sum_{k,l} f[k,l]\exp\left(-\frac{1}{2}\left[\frac{(m-k)^2}{\sigma_m^2}+\frac{(n-l)^2}{\sigma_n^2}\right]\right)\exp\left(-\frac{1}{2}\frac{(f[m,n]-f[k,l])^2}{\rho^2}\right)$$

C is normalization constant.

- Harmonic filter:

$$g[m,n] = \frac{N}{\sum\limits_{[k,l]\in S_{mn}} \frac{1}{f[k,l]+\epsilon}}$$

- Contra Harmonic filter:

$$g[m,n] = \frac{\sum\limits_{[k,l]\in S_{mn}} f[k,l]^{D+1}}{\sum\limits_{[k,l]\in S_{mn}} f[k,l]^{D}}$$

for $D = 0$, this is the arithmetic mean filter.

Rank Filtering

- Median filter:

$$g[m,n] = \underset{[k,l]\in S_{mn}}{\text{median}}\ f[k,l]$$

For example:

$$\text{median}\left\{\begin{bmatrix}1 & 2 & 3\\3 & 4 & 5\\5 & 6 & 7\end{bmatrix}\right\} = \text{median}\left\{\begin{bmatrix}7 & 6 & 5 & 5 & 4 & 3 & 3 & 2 & 1\end{bmatrix}\right\} = 4$$

- max and min filters:

$$g[m,n] = \max_{[k,l]\in S_{mn}} f[k,l] \qquad g[m,n] = \min_{[k,l]\in S_{mn}} f[k,l]$$

For example: For example:

$$\max\left\{\begin{bmatrix}1 & 2 & 3\\3 & 4 & 5\\5 & 6 & 7\end{bmatrix}\right\} = 7 \qquad \min\left\{\begin{bmatrix}1 & 2 & 3\\3 & 4 & 5\\5 & 6 & 7\end{bmatrix}\right\} = 1$$

- Midpoint filter:

$$g[m,n] = \frac{1}{2}\left[\max_{[k,l]\in S_{mn}} f[k,l] + \min_{[k,l]\in S_{mn}} f[k,l]\right]$$

- d-Trimmed mean filter:

$$g[m,n] = \frac{1}{N-2d}\sum_{[k,l]\in S_{mn}} f[k,l]$$

For example:

$$\begin{aligned}\text{2-Trimmed Mean}\left\{\begin{bmatrix}1 & 2 & 3\\3 & 4 & 5\\5 & 6 & 7\end{bmatrix}\right\} &= \text{2-Trimmed Mean}\left\{\begin{bmatrix}7 & 6 & 5 & 5 & 4 & 3 & 3 & 2 & 1\end{bmatrix}\right\}\\ &= \text{2-Trimmed Mean}\left\{\begin{bmatrix}5 & 5 & 4 & 3 & 3\end{bmatrix}\right\}\\ &= \frac{1}{9-4}\left[5+5+4+3+3\right]\end{aligned}$$

Noise Models:

- Gaussian Noise:

$$p(\nu) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\nu-\mu)^2}{2\sigma^2}\right)$$

- Laplacian noise:

$$p(\nu) = \frac{1}{2b} \exp\left(-\frac{|\nu-\mu|}{b}\right)$$

- Generalized noise:

$$p(\nu) = \frac{1}{2b\,\Gamma\left(1+\frac{1}{\alpha}\right)} \exp\left(-\frac{|\nu-\mu|}{b}\right)^{\alpha}$$

$\Gamma(x)$ is Gamma function and $b = \sigma\sqrt{\frac{\Gamma(1/\alpha)}{\Gamma(3/\alpha)}}$

Wiener Filter

For the model $g[m,n] = h[m,n] * f[m,n] + \eta[m,n]$, the Wiener is defined as:

$$W(\mu,\nu) = \frac{S_{fg}(\mu,\nu)}{S_{gg}(\mu,\nu)}$$

Where, S_{fg} cross-spectral density of f and g, which is the Fourier transform of the cross-correlation. The cross-correlation is defined as:

$$R_{fg}[m,n] = \mathbb{E}\left\{f[r+m, s+n]g^*[r,s]\right\}$$